Solution of the Black-Scholes Equation for Pricing of Barrier Option

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In this paper two different methods are presented to approximate the solution of the Black-Scholes equation for valuation of barrier option. These techniques can be applied directly for all types of differential equations, homogeneous or inhomogeneous. The use of these methods provides the solution of the problem in a closed form while the mesh point techniques provide the approximation at mesh points only. Also, the two schemes need less computational work in comparison with the traditional methods. These techniques can be employed for problems with initial condition. In this paper we use the variational iteration and homotopy perturbation methods for solving the Black-Scholes equation with terminal condition. Numerical results are compared with theoretical solutions in order to confirm the validity of the presented procedures.

Key words: Black-Scholes Equation; Barrier Option; Variational Iteration Method (VIM); Homotopy Perturbation Method (HPM); Semi-Analytic Approach; Stochastic Differential Equation (SDE); Mathematical Finance.
AMS Subject Classifications: 65M99

1. Introduction

The pricing of options is an important subject in the quantitative finance. It is of both theoretical and practical importance since the use of options thrives in the finance industry recently [1]. As said in [2] the interest in pricing financial derivative – including pricing options – arises from the fact that financial derivatives can be used to minimize losses caused by price fluctuations of the underlying assets [2]. In the early 1970s Fischer Black and Myron Scholes made an important discovery in the pricing of stock options. This involved the development of what has become known as the Black-Scholes model [3, 4]. Now we follow [4] to present the assumptions in the following form:

1. The price of asset follows a geometric Brownian motion $W(t)$, meaning that s satisfies the following stochastic differential equation:

$$\mathrm{d}s(t) = \mu s(t)\mathrm{d}t + \sigma s(t)\mathrm{d}W(t).$$

2. The trend or drift μ (measures the average of growth of the asset price) and volatility σ (measures the standard deviation of the returns) are constant for $0 \le t \le T$.

3. There are no transactions costs or taxes. All securities are perfectly divisible, i. e. the market is frictionless.

4. There are no dividends on the stock during the life of the option.

5. There are no riskless arbitrage opportunities.

6. Security trading is continuous.

7. Investors can borrow or lend at the same risk-free rate of interest.

8. The short-term risk-free rate of interest r is constant.

Also we refer the interested reader to [2, 3, 5]. Some of these assumptions have been relaxed by other researchers. For example variations on the Black-Scholes formula can be used when r and σ are functions of time [4, 6]. Some of the option prices can be determined by analytically solving the so-called Black-Scholes equation. The exact solution of barrier option is obtained in [7]. Since the other option prices can not be found analytically, many researchers have used the numerical schemes to find the solution of the Black-Scholes equation [2, 6, 8 – 10]. As is said in [11] some kind of analytical techniques for valuation of the American options have been proposed by Barone-

Adesi [12], MacMillan [13] and Johnson [14]. These techniques basically transform the free boundary value problem to an integral equation whose analytical formula is obtained by either assuming a numerical approximation of the unknown optimal exercise boundary or a polynomial expansion of the unknown integrand [11]. Cortes and his co-authors in [8] proposed a method based on the Mellin transform for solving the Black-Scholes matrix equation. Company and his co-authors [9] used a delta-defining sequence of the involved generalized Dirac delta function and applied the Mellin transform and obtained the solution of the modified Black-Scholes equation with jump conditions for discrete dividend payments and then they used the Gauss-Hermite approach and the composite Simpson's rule for numerical approximations. Authors of [15] by using the Mellin transform of a class of weak functions obtained a candidate integral formula for the solution of the problem and then proved it was a rigorous solution. Recently, Ballester and his co-authors [16] by applying a semidiscretization technique on the asset, proposed a numerical solution for the partial differential equation modelling option with a discrete dividend payment. Finite element approximations are employed in [17] to solve this equation. A semi-discrete Galerkin formulation combined with high-order Lagrangian finite elements was used to solve this equation [18]. Binomial methods, integral equation method, penalty methods, and Monte-Carlo technique is employed in [19 – 22], respectively. Lattice approach is introduced to value path dependent options in [23]. If an option is constructed with many assets, huge computer memory is necessary in the penalty method. The binomial and lattice methods also have similar features as the penalty method. Therefore, if the option is constructed with many assets, the Monte-Carlo method is more effective, but the computational accuracy of the Monte-Carlo method depends on the quality of the algorithm to generate random numbers [6]. Another popular method is the projected successive overrelaxation method (PSOR) [5] but the iterative procedure converges slowly. Hon and Mao in [10] utilized the multi-quadric method to solve this equation. They proposed the global radial basis functions [6], particularly Hardly's multi-quadratic, as a special approximation for the numerical solution of the option value and its derivatives in the Black-Scholes equation. They transformed the Black-Scholes equation into a system of first-order equations in time. Thus the American options can then be solved by using the fourth-order Runge-Kutta formula [24]. Authors of [25] solved this equation by the linear programming method. Authors of [26] used the homotopy perturbation method to solve stochastic equations. We refer the interested reader to the books [4, 5, 7] for more information about this equation.

The approach in the current paper is different as we use the semi-analytical techniques [27, 28].

2. The Main Approach (A Semi-Analytic Approach)

In this paper we use the variational iteration method (VIM) and the homotopy perturbation method (HPM) which were proposed by the Chinese researcher J. H. He [29, 30]. These methods have been employed to solve a large class of problems with approximations converging rapidly to accurate solutions [31]. VIM is employed in [32] to solve the parabolic inverse problem. He employed VIM for solving the Duffing equation, the problem of mathematical pendulum, and the equation of eardrum vibrations [31]. Also He applied VIM to solve fractional differential equations [33]. VIM is an effective method for searching for various wave solutions including periodic solutions, solitons, and compacton solutions without linearization or weak nonlinearity assumptions [34]. He applied VIM to autonomous ordinary differential systems [35] and nonlinear equations with convolution product nonlinearity [36]. Wazwaz [37] used VIM to determine rational solutions for the Korteweg-de Vries (KdV), K(2,2), Burgers, and cubic Boussinesq equations. In [38] VIM is compared with the Adomian decomposition method for homogeneous and nonhomogeneous advection problems. The Fokker-Planck equation is solved in [39] using the variational iteration method. This method is employed in [40] to solve a model which describes the biological species living together. Also a partial integro-differential equation which arises in the modelling of the heat conduction in materials with memory is investigated in [41]. This technique is used in [42] to solve an initial-boundary value problem that combines Neumann and integral conditions for the wave equation. Author of [43] employed this method to solve the nonlinear Goursat problem. The VIM is investigated in [44] to solve some problems in calculus of variations. The variation iteration method is used to solve the Kawahara equation arising in the modelling of water waves [45]. The convergence of VIM is investigated in [46].

Our second procedure in the semi-analytical approach is based on the homotopy perturbation method. Authors of [47] used HPM to approximate the generalized Emden-Fowler equations. HPM was applied to solve the Klein-Gordon and Sine-Gordon equations [48]. Odibat and Momani [49] applied HPM to solve quadratic Riccati differential equation of fractional order. Ramos in [50] presented HPM for the solution of the Lane-Emden equation which provides series solutions to this equation. Authors of [51] used an application of the multistage homotopy perturbation method for the solutions of the Chen system. Belendez and his co-authors in [52] applied HPM for an antisymmetric nonlinear oscillator. Authors of [53] investigated the solution of nonlinear integral equations and employed the homotopy perturbation method to solve them. This method is modified in [54] to compute the periodic solutions of a nonlinear oscillator with discontinuities for which the elastic force term is proportional to $\operatorname{sgn}(x)$. It is modified by truncating the infinite series corresponding to the first-order approximate solution before introducing this solution in the second-order linear differential equation. Delay differential equations are solved in [55] using the homotopy perturbation method. The HPM is proposed in [56] to solve the inverse problem of the diffusion equation. Also a nonlinear system of differential equations is solved in [57] using this method. HPM was successfully applied to many other problems. The interested reader can see the references [27, 28, 58 – 63]. The remaining of this paper is organized as follows. In Section 3, the option contracts are described. Variational iteration method, homotopy perturbation method and their numerical results are presented in Sections 4 and 5, respectively. Finally, the conclusions are summarized in Section 6.

3. Different Types of Options

As is said in [4] there are two basic types of options. A call option gives the holder of the option the right (not the obligation) to buy an asset by a certain date for a certain price. A put option gives the holder the right (not the obligation) to sell an asset by a certain date for a certain price. The date specified in the contract is known as the expiration date or maturity. The price specified in the contract is known as the exercise price or strike price [3, 4]. Options can be either American or European, American options can be exercised at any time during the life of the option, whereas European options can be exercised exactly at the maturity [5, 7].

In this paper, we will focus on the barrier options. These options are only weekly path-dependent (an option whose payoff at exercise or expiry depends, in some non-trivial way, on the past history of the underlying asset price as well as its price at exercise or expiry) and satisfy the Black-Scholes equation. Calls or puts barrier options are categorized as follows [6, 7]:

1. up-and-in: the option expires worthless unless the barrier $s = B$ is reached from below before expiry;
2. down-and-in: the option expires worthless unless the barrier $s = B$ is reached from above before expiry;
3. up-and-out: the option expires worthless if the barrier $s = B$ is reached from below before expiry;
4. down-and-out: the option expires worthless if the barrier $s = B$ is reached from above before expiry.

Here, we shall consider the down-and-out option that is constructed with only one asset. Let s be the current stock price, σ the volatility of the stock, and r the risk-free interest rate. Let V be the price of a barrier option at time t with expiration date T, exercise price E, and barrier B. As is mentioned in [6] this option becomes invalid if the asset price s reaches the barrier B from above the barrier during the day of purchase and the expiration date. Unless the asset price s reaches the barrier B ($s > B$) the option is an European call option [6]. The option price of the down-and-out option is governed with the Black-Scholes equation [5 – 7]

$$\frac{\partial V(s,t)}{\partial t} + \frac{\sigma^2}{2} s^2 \frac{\partial^2 V(s,t)}{\partial s^2} + rs\frac{\partial V}{\partial s} - rV = 0, \text{ if } s > B,$$
$$V = 0, \text{ if } s \le B, \tag{1}$$

where σ and r are constant. The terminal condition on T is given as

$$V(s,T) = \max(s(T) - E, 0). \tag{2}$$

If s reaches B, the option is invalid, i. e. $V(B,t) = 0$.

We refer the interested reader to [2] for more discussion.

4. Variational Iteration Method

To illustrate the basic idea of the variational iteration method [29, 31, 32, 34, 35, 37 – 44], we consider the following equation:

$$\mathbb{L}V(s,t) + \mathbb{N}V(s,t) = g(s,t), \quad V(s,0) = f(s), \tag{3}$$

where $\mathbb{L}$ is a linear operator, $\mathbb{N}$ is a nonlinear operator, and $g(s,t)$ is an inhomogeneous term. Then using the

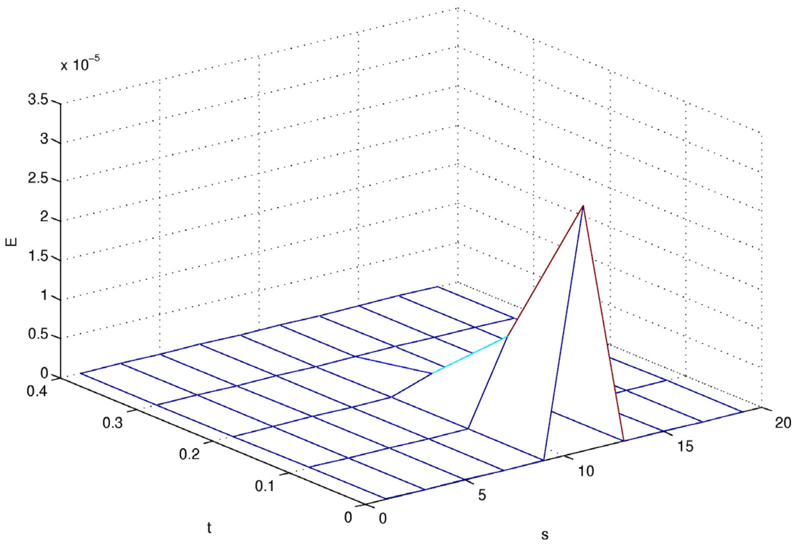

Fig. 1 (colour online). Plot of absolute error by VIM.

variational iteration method, the following correction functional is constructed:

$$V_{n+1} = V_n + \int_0^t \lambda(\xi)(\mathbb{L}V_n + \mathbb{N}\tilde{V}_n - g)\mathrm{d}\xi, \quad (4)$$

where λ is the Lagrange multiplier which can be identified optimally via variational theory. The subscript n denotes the nth approximation, $\tilde{V}_n$ is a restricted variation. Making the correction functional (4) stationary [30, 64] and noticing that $\delta(0) = 0$, λ can be identified. For linear problems its exact solution can be obtained by only one iteration step due to the fact that the Lagrange multiplier can be exactly identified. In nonlinear problems, in order to determine the Lagrange multiplier in a simple manner, the nonlinear terms have to be considered as restricted variations [31]. For applying VIM on (1)–(2), we have proposed this method in the following form:

$$V_{n+1} = V_n + \int_t^T \lambda(\xi)\left(\frac{\partial V_n}{\partial \xi} + \frac{\sigma^2}{2}s^2\frac{\partial^2\tilde{V}_n}{\partial s^2} + rs\frac{\partial \tilde{V}_n}{\partial s} - rV_n\right)\mathrm{d}\xi. \quad (5)$$

Taking the variation from both sides of (5) with respect to V_n, we have

$$\delta V_{n+1} = \delta V_n + \delta\int_t^T \lambda(\xi)\left(\frac{\partial V_n}{\partial \xi} + \frac{\sigma^2}{2}s^2\frac{\partial^2\tilde{V}_n}{\partial s^2} + rs\frac{\partial \tilde{V}_n}{\partial s} - rV_n\right)\mathrm{d}\xi = \delta V_n + \int_t^T \lambda(\xi)\delta V_n'\mathrm{d}\xi - \int_t^T r\lambda(\xi)\delta V_n\mathrm{d}\xi = 0.$$

Using integration by parts and considering $\delta V_n(T) = 0$, we can write

$$\delta V_{n+1} = \delta V_n - \lambda(\xi)\delta V_n|_{\xi=t} - \int_t^T \lambda'(\xi)\delta V_n\mathrm{d}\xi - \int_t^T r\lambda(\xi)\delta V_n\mathrm{d}\xi = 0,$$

which yields the following stationary conditions:

$$1 - \lambda(\xi)|_{\xi=t} = 0, \quad (6)$$

$$\lambda'(\xi) + r\lambda(\xi) = 0. \quad (7)$$

Table 1. Comparison between the analytical and VIM solutions for the down-and-out option, $E = 10$, $B = 9$, $\sigma = 0.05$, $r = 0.05$, and $t = 0$.

s	Analytical	VIM	Absolute error
$T = 0.25, 0.5$			
1	0	0	0
3	0	0	0
5	0	0	0
7	0	0	0
9	0	0	0
$T = 0.25$			
11	1.12422244	1.12422199	$4.456863 \cdot 10^{-7}$
13	3.12422199	3.12422199	$8.88178 \cdot 10^{-16}$
15	5.12422199	5.12422199	$8.88178 \cdot 10^{-16}$
17	7.12422199	7.12422199	$2.664535 \cdot 10^{-15}$
19	9.12422199	9.12422199	$1.776357 \cdot 10^{-15}$
RMSE $= 4.456863 \cdot 10^{-8}$			
$T = 0.5$			
11	1.24693225	1.24690087	$3.13765 \cdot 10^{-5}$
13	3.24690087	3.24690087	$4.441 \cdot 10^{-16}$
15	5.24690087	5.24690087	$1.7764 \cdot 10^{-15}$
17	7.24690087	7.24690087	$1.7764 \cdot 10^{-14}$
19	9.24690087	9.24690087	0
RMSE $= 3.137655 \cdot 10^{-6}$			

Therefore we have $\lambda(\xi) = \exp(-r(\xi - t))$. We find the following iteration formula:

$$V_{n+1} = V_n + \int_t^T \exp(-r(\xi - t))\left(\frac{\partial V_n}{\partial \xi} + \frac{\sigma^2}{2} s^2 \frac{\partial^2 \tilde{V}_n}{\partial s^2} + rs\frac{\partial \tilde{V}_n}{\partial s} - rV_n\right) \mathrm{d}\xi. \quad (8)$$

We take the initial approximation $V_0 = V(s,T) = \max(s(T) - E, 0) = \frac{|s(T)-E|+(s(T)-E)}{2}$. The next iteration can easily be obtained from (8). The numerical results are shown in Table 1 and Figure 1. The root-mean-square-error (RMSE) defined by

$$\mathrm{RMSE} = \frac{1}{n}\sqrt{\sum_{i=1}^{n}(V_{\mathrm{VIM}}(s_i,0) - V_{\mathrm{Analytical}}(s_i,0))^2}, \quad (9)$$

in which s_i are the stock values in Table 1, is computed for $T = 0.25$ and $T = 0.5$.

It is worth to point out that the variational iteration algorithms were summarized in the following review article [65]. Also note that the variational iteration algorithm presented in this paper is the variational iteration algorithm-I, there are other algorithms, i. e., variational iteration algorithm-II and variational iteration algorithm-III, which can be used.

5. Homotopy Perturbation Method

To illustrate the basic concepts of the homotopy perturbation method [47 – 52, 54 – 57], we consider the following equation:

$$\mathbb{L}(V) + \mathbb{N}(V) - g(r) = 0, \quad r \in \Omega,$$

with boundary condition

$$B\left(V, \frac{\partial V}{\partial n}\right) = 0, \quad r \in \Gamma,$$

where $\mathbb{L}$ and $\mathbb{N}$ are linear and nonlinear operators, respectively, B is a boundary operator, $g(r)$ is a known analytical function, and Γ is the boundary of the domain Ω. Homotopy perturbation structure is

$$\begin{aligned} H(v,p) &= (1-p)[\mathbb{L}(v) - \mathbb{L}(v_0)] \\ &\quad + p[\mathbb{L}(v) + \mathbb{N}(v) - g(r)] = 0, \quad (10) \\ v(r,p) &: \Omega \times [0,1] \longrightarrow \mathbb{R}, \end{aligned}$$

where p is an embedding parameter and v_0 is the first approximation that satisfies the boundary condition.

Table 2. Comparison between the analytical and HPM solutions for the down-and-out option, $E = 10$, $B = 9$, $\sigma = 0.05$, $r = 0.05$, and $t = 0$.

s	Analytical	HPM	Absolute error
$T = 0.25, 0.5$			
1	0	0	0
3	0	0	0
5	0	0	0
7	0	0	0
9	0	0	0
$T = 0.25$			
11	1.12422244	1.12421875	$3.690747 \cdot 10^{-6}$
13	3.12422199	3.12421875	$3.245061 \cdot 10^{-6}$
15	5.12422199	5.12421875	$3.245061 \cdot 10^{-6}$
17	7.12422199	7.12421875	$3.245061 \cdot 10^{-6}$
19	9.12422199	9.12421875	$3.245061 \cdot 10^{-6}$
$\mathrm{RMSE} = 7.466143 \cdot 10^{-7}$			
$T = 0.5$			
11	1.24693225	1.24687500	$5.725627 \cdot 10^{-5}$
13	3.24690087	3.24687500	$2.587971 \cdot 10^{-5}$
15	5.24690087	5.24687500	$2.587971 \cdot 10^{-5}$
17	7.24690087	7.24687500	$2.587971 \cdot 10^{-5}$
19	9.24690087	9.24687499	$2.587971 \cdot 10^{-5}$
$\mathrm{RMSE} = 7.718367 \cdot 10^{-6}$			

As p increases from 0 to 1 the solution of (10) varies from the initial guess v_0 to the solution v. Expanding v in Taylor series with respect to the embedding parameter p, we have

$$v = v_0 + pv_1 + p^2 v_2 + \dots. \quad (11)$$

The series (11) converges at $p = 1$ and is discussed in [29]. As an approximation for the solution we consider

$$V = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \dots. \quad (12)$$

For applying HPM on (1) and (2), we use a simple transformation $\tau = T - t$:

$$\begin{aligned} &\frac{\partial V}{\partial \tau} - \frac{\sigma^2}{2} s^2 \frac{\partial^2 V}{\partial s^2} - rs\frac{\partial V}{\partial s} + rV = 0, \text{ if } s > B, \\ &V = 0, \text{ if } s \le B, \\ &V(s,0) = \max(s(T) - E, 0). \end{aligned} \quad (13)$$

From (10) we have

$$H(v,p) = \frac{\partial V}{\partial \tau} - \frac{\sigma^2}{2} p^2 s^2 \frac{\partial^2 V}{\partial s^2} - rps\frac{\partial V}{\partial s} + rpV = 0. \quad (14)$$

Substituting v from (12) into (13) and rearranging based on powers of p-terms, we have:

$$\frac{\partial v_0}{\partial \tau} = 0, \quad (15)$$

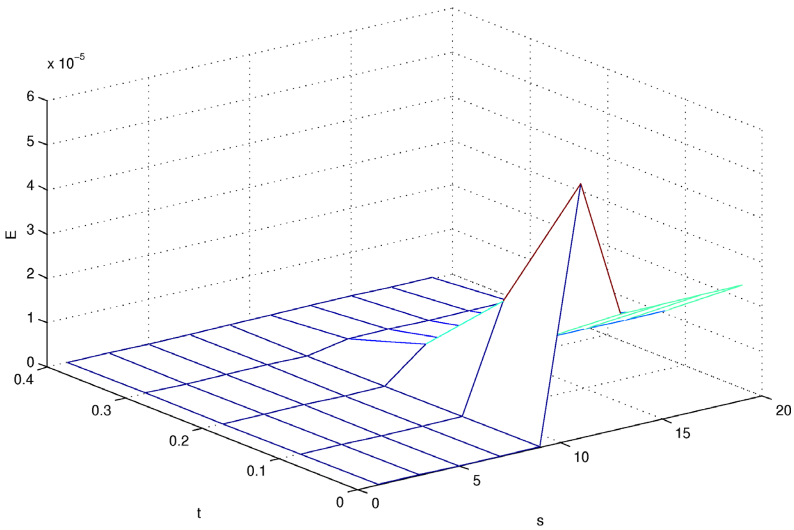

Fig. 2 (colour online). Plot of absolute error by HPM.

$$\frac{\partial v_1}{\partial \tau} - \frac{\sigma^2}{2} s^2 \frac{\partial^2 v_0}{\partial s^2} - rs\frac{\partial v_0}{\partial s} + rv_0 = 0, \tag{16}$$

$$\frac{\partial v_2}{\partial \tau} - \frac{\sigma^2}{2} s^2 \frac{\partial^2 v_1}{\partial s^2} - rs\frac{\partial v_1}{\partial s} + rv_1 = 0, \tag{17}$$

therefore we can write

$$v_0(s,\tau) = s - E, \tag{18}$$

$$v_1(s,\tau) = rE\tau, \tag{19}$$

$$v_2(s,\tau) = -\frac{r^2}{2}E\tau^2. \tag{20}$$

The solution of (13) when $p \to 1$ will be as follows:

$$\begin{aligned} V(s,\tau) &= v_0(s,\tau) + v_1(s,\tau) + v_2(s,\tau) \\ &= s + E\left(-1 + r\tau - \frac{r^2}{2}\tau^2\right). \end{aligned} \tag{21}$$

The numerical results are shown in Table 2 and Figure 2.

It is worth to note that in the standard Black-Scholes model of option pricing, the volatility is assumed to be known. This strong assumption is incompatible with that of the market price movement. As is said in [66] in fact, the phenomenon so-called volatility skew or smile exists in all the major stock index markets today. Very recently authors of [66] investigated the solution for uncertain volatility model in option pricing. They proposed a computational technique to solve it. Their method is based on a fitted finite volume method. We would like to mention that for the algorithm, the choice of initial solution is of utter importance, we refer the interested reader to [67, 68].

6. Conclusions

In this work we have studied the barrier option with variational iteration and homotopy perturbation methods. Application of these methods is easy and calculation of approximations is direct and straight forward. The two new schemes provide the solution of the problem in a closed from while the mesh point techniques [2, 69] provide the approximation at mesh point only. Both VIM and HPM obtain accurate approximation for the given problem. Also the presented methods do not provide any linear or nonlinear system of equations, thus these procedures reduce the volume of calculations by not requiring the solution of linear or nonlinear systems.

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